

To André Haefliger with admiration

Holonomic approximation and Gromov's h -principle

Y. M. Eliashberg

Stanford University, USA

N. M. Mishachev

Lipetsk Technical University, Russia

and Stanford University, USA

Abstract

In 1969 M. Gromov in his PhD thesis [G1] greatly generalized Smale-Hirsch-Phillips immersion-submersion theory (see [Sm],[Hi], [Ph]) by proving what is now called the h -principle for invariant open differential relations over open manifolds. Gromov extracted the original geometric idea of Smale and put it to work in the maximal possible generality. Gromov's thesis was brought to the West by A. Phillips and was popularized in his talks. However, most western mathematicians first learned about Gromov's theory from A Haefliger's article [H]. The current paper is devoted to the same subject as the papers of Gromov and Haefliger. It seems to us that we further purified Smale-Gromov's original idea by extracting from it a simple but very general theorem about holonomic approximation of sections of jet-bundles (see Theorem 1.2.1 below). We show below that Gromov's theorem as well as some other results in the h -principle spirit are immediate corollaries of Theorem 1.2.1. ¹

1 Holonomic approximation

1.1 Jets and holonomy

Given a C^∞ -smooth fibration $p : X \rightarrow V$, we denote by $X^{(r)}$ the space of r -jets of smooth sections $f : V \rightarrow X$ and by $J_f^r : V \rightarrow X^{(r)}$ the r -jet of a section $f : V \rightarrow X$. When the fibration $X = V \times W \rightarrow V$ is trivial then the space $X^{(r)}$ is sometimes denoted by $J^r(V, W)$, and called the space of r -jets of maps $V \rightarrow W$. A section $F : V \rightarrow X^{(r)}$ is called *holonomic* if it has the form J_f^r for a section $f : V \rightarrow X$. The correspondence $f \mapsto J_f^r$ defines the derivation map

¹The authors are partially supported by the National Science Foundation

$J^r : \text{Sec } X \rightarrow \text{Sec } X^{(r)}$. Its one-to-one image $J^r(\text{Sec } X)$ coincides with the space $\text{Hol} X^{(r)} \subset \text{Sec } X^{(r)}$ of holonomic sections, i.e. we have

$$\text{Sec } X \xrightarrow{J^r} \text{Hol } X^{(r)} \hookrightarrow \text{Sec } X^{(r)}.$$

Notice that the C^0 -topology on $\text{Sec } X^{(r)}$ induces via J^r the C^r -topology on $\text{Sec } X$.

Following Gromov's book [G2] we will denote by $\mathcal{O}p A$ an arbitrary small but non-specified (open) neighborhood of a subset $A \subset V$. We will assume that the manifold V and the bundle $X^{(r)}$ are endowed with Riemannian metrics and denote by $U_\varepsilon(A)$ the metric ε -neighborhood $\{x \mid \text{dist}(x, A) < \varepsilon\}$.

Given an arbitrary subset $A \subset V$ a section $F : A \rightarrow X^{(r)}$ is called *holonomic* if there exists a holonomic section $\tilde{F} : \mathcal{O}p A \rightarrow X^{(r)}$ such that $\tilde{F}|_A = F$. A section $F : V \rightarrow X$ is called *holonomic over* $A \subset V$ if the restriction $F|_A$ is holonomic. Given a fibration $\pi : V \rightarrow B$ we say that a section $F : V \rightarrow X^{(r)}$ is *fiberwise holonomic* if there exists a continuous family of holonomic sections $\tilde{F}_b : \mathcal{O}p \pi^{-1}(b) \rightarrow X^{(r)}$, $b \in B$, such that for each $b \in B$ the sections \tilde{F}_b and F coincide over the fiber $\pi^{-1}(b)$. Note the following trivial but important fact:

1.1.1 *Any section $F : V \rightarrow X$ is holonomic over any point $v \in V$. Moreover, it is fiberwise holonomic with respect to the trivial fibration $\text{Id} : V \rightarrow V$.*

Indeed, we can take the Taylor polynomial map which corresponds to $F(v)$ with respect to some local coordinate system centered at v as a section $\tilde{F}_v : \mathcal{O}p v \rightarrow X^{(r)}$. Moreover, the local coordinate system can be chosen smoothly depending on its center. ►

1.2 Holonomic approximation

Question: *Is it possible to approximate any section $F : V \rightarrow X^{(r)}$ by a holonomic section? In other words, given a r -jet section and an arbitrary small neighborhood of the image of this section in the jet space, can one find a holonomic section in this neighborhood?*

The answer is evidently negative (excluding, of course, the situation when the initial section is already holonomic). For instance, in the case $r = q = 1$ the question has the following geometrical reformulation: given a function and a n -plane field along the graph of this function, can one C^0 -perturb this graph to make it almost tangent to the given field?

The problem of finding a holonomic approximation of a section of the r -jet space *near a submanifold* $A \subset \mathbb{R}^n$ is also usually unsolvable. The only exception is the zero-dimensional case: as we already stated above in 1.1.1 any section can be approximated near any point by the r -jet of the respective Taylor polynomial map.

In contrast, the following theorem says that we *always can find* a holonomic approximation of a section $F : V \rightarrow X^{(r)}$ *near a slightly deformed* submanifold $\tilde{A} \subset V$ if the original set $A \subset V$ is of positive codimension.

1.2.1 (Holonomic Approximation Theorem) *Let $A \subset V$ be a polyhedron of positive codimension and $F : \mathcal{O}p A \rightarrow X^{(r)}$ be a section. Then for arbitrary small $\delta, \varepsilon > 0$ there exist a diffeomorphism $h : V \rightarrow V$ with $\|h - \text{Id}\|_{C^0} < \delta$ and a holonomic section $\tilde{F} : \mathcal{O}p h(A) \rightarrow X^{(r)}$ such that the image $h(A)$ is contained in the domain of the definition of the section F and*

$$\|\tilde{F} - F|_{\mathcal{O}p h(A)}\|_{C^0} < \varepsilon.$$

We use the term *polyhedron* here in the sense that A is a subcomplex of a certain smooth triangulation of the manifold V .

As we will see below, the *relative* and the *parametric* versions of the theorem are also true. In the relative version the section F is assumed to be already holonomic over $\mathcal{O}p B$, where B is a subpolyhedron of A , while the diffeomorphism h is constructed to be fixed on $\mathcal{O}p B$ and \tilde{F} is required to coincide with F on $\mathcal{O}p B$. Here is the parametric version of 1.2.1.

1.2.2 (Parametric Holonomic Approximation Theorem) *Let $A \subset V$ be a polyhedron of positive codimension and $F_z : \mathcal{O}p A \rightarrow X^{(r)}$ be a family of sections parametrized by a cube I^m , $m = 0, 1, \dots$. Suppose that the sections F_z are holonomic for $z \in \mathcal{O}p \partial I^m$. Then for arbitrary small $\delta, \varepsilon > 0$ there exist a family of diffeomorphisms $h_z : V \rightarrow V$ and a family of holonomic section $\tilde{F}_z : \mathcal{O}p h_z(A) \rightarrow X^{(r)}$, $z \in I^m$, such that*

- $\|h_z - \text{Id}\|_{C^0} < \delta$;
- $h_z = \text{Id}$ for $z \in \mathcal{O}p \partial I^m$;
- $\|\tilde{F}_z - F_z|_{\mathcal{O}p h_z(A)}\|_{C^0} < \varepsilon$;
- $\tilde{F}_z = F_z$ for $z \in \mathcal{O}p \partial I^m$.

Using the induction over skeleta of the polyhedron A , and taking into account that the fibration $X \rightarrow V$ is trivial over simplices, we reduce Theorem 1.2.1 to its special case for the pair $(A, B) = (I^k, \partial I^k) \subset \mathbb{R}^n$. We consider this special case in the next section.

1.3 Holonomic Approximation over a cube

1.3.1 (Holonomic Approximation over a cube) *Let $I^k \subset \mathbb{R}^n$, $k < n$, be the unit cube in the coordinate subspace $\mathbb{R}^k \subset \mathbb{R}^n$ of the first k coordinates. Then for any section*

$$F : \mathcal{O}_p I^k \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

which is holonomic over $\mathcal{O}_p \partial I^k$ and for arbitrary small $\delta, \varepsilon > 0$ there exist a diffeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_k)),$$

and a holonomic section

$$\tilde{F} : \mathcal{O}_p h(I^k) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that

- $|\varphi| < \delta$; $\varphi|_{\mathcal{O}_p \partial I^k} = 0$;
- the image $h(I^k)$ is contained in the domain of the definition of the section F ;
- $\tilde{F}|_{\mathcal{O}_p \partial I^k} = F|_{\mathcal{O}_p \partial I^k}$ and
- $\|\tilde{F} - F\|_{\mathcal{O}_p h(I^k)}|_{C^0} < \varepsilon$.

Theorem 1.3.1 will be deduced from Inductional lemma 1.3.2 which we formulate below.

Let $\pi_l : I^k \rightarrow I^{k-l}$, $l = 1, \dots, k$, be the projection

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_{k-l})$$

whose fibers are l -dimensional cubes

$$I^l(y) = y \times I^l, \quad y = (x_1, \dots, x_{k-l}) \in I^{k-l}.$$

Note that for $l < k$ we have $I^l(y) = \bigcup_{t \in I} I^{l-1}(y, t)$ where $t = x_{k-l+1}$.

1.3.2 (Inductional lemma) *Suppose that a section*

$$F : \mathcal{O}_p I^k \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

is holonomic over $\mathcal{O}_p \partial I^k$ and for a positive integer $l \leq k$ fiberwise holonomic with respect to the fibration $\pi_{l-1} : I^k \rightarrow I^{k-l+1}$. Then for arbitrary small $\delta, \varepsilon > 0$ there exist a diffeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_k)),$$

and a section

$$\tilde{F} : \mathcal{O}_p h(I^n) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q),$$

such that

- $|\varphi| < \delta$; $\varphi|_{\mathcal{O}_p \partial I^k} = 0$;
- the image $h(I^k)$ is contained in the domain of the definition of the section F ;
- $\tilde{F}|_{\mathcal{O}_p \partial I^k} = F|_{\mathcal{O}_p \partial I^k}$;
- $\|\tilde{F} - F\|_{\mathcal{O}_p h(I^k)}|_{C^0} < \varepsilon$;
- the section $\tilde{F}|_{h(I^k)}$ is fiberwise holonomic with respect to the fibration

$$\pi_l \circ h^{-1} : h(I^k) \rightarrow I^{k-l}.$$

Proof of Inductional lemma: We recommend to the reader to keep in mind the simplest case $n = 2$, $k = l = 1$ while reading the proof for the first time.

For $(y, t) \in I^{k-l} \times I$ set

$$U_\delta(y, t) = U_\delta(I^{l-1}(y, t)) \quad \text{and} \quad U_\delta^\partial(y, t) = U_\delta(\partial I^{l-1}(y, t)).$$

It follows from the definition of a fiberwise holonomic section and compactness arguments that we can choose $\delta > 0$ so small that there would exist a continuous family of holonomic sections

$$F_{y,t} = J_{f_{y,t}}^r : U_\delta(y, t) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q), \quad (y, t) \in I^{k-l+1} = I^{k-l} \times I,$$

such that for all $(y, t) \in I^{k-l} \times I$:

- $F_{y,t}$ is defined on $U_\delta(y, t)$;
- $F_{y,t}|_{I^{l-1}(y,t)} = F|_{I^{l-1}(y,t)}$.

We can further adjust δ and the family $F_{y,t}$ in order to have

- $F_{y,t}|_{U_\delta^\partial(y,t)} = F|_{U_\delta^\partial(y,t)}$ for all $(y, t) \in I^{k-l} \times I$ and
- $F_{y,t} = F|_{U_\delta(y,t)}$ for all $(y, t) \in \mathcal{O}p \partial(I^{k-l} \times I)$.

For a sufficiently large N , which is determined by the Interpolation Property below, we set

$$\begin{aligned} U_i(y) &= U_\delta(y, \frac{i}{2N}), \quad U_i^\partial(y) = U_\delta^\partial(y, \frac{i}{2N}), \quad y \in I^{k-l}, \quad i = 0, \dots, 2N, \\ U_i^{\text{top}}(y) &= U_i(y) \cap \{x_n \geq \frac{\delta}{2}\}, \\ U_i^{\text{bot}}(y) &= U_i(y) \cap \{x_n \leq -\frac{\delta}{2}\}. \end{aligned}$$

We will write F_y^i and f_y^i instead of $F_{y, \frac{i}{N}}$ and $f_{y, \frac{i}{N}}$.

Notice that

$$\sigma(N) = \max_{y \in I^{k-l}, i=1, \dots, 2N, x \in U_i(y) \cap U_{i-1}(y)} \|F_y^i(x) - F_y^{i-1}(x)\| \xrightarrow{N \rightarrow \infty} 0,$$

and hence we have the following

Interpolation Property. *For any $\varepsilon > 0$, a sufficiently large N and all odd integers $i = 1, 3, \dots, 2N-1$ there exist continuous families of holonomic sections*

$$G_y^i = J_{g_y^i}^r : U_i(y) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q), \quad y \in I^{k-l},$$

such that

- G_y^i interpolate between F_y^{i-1} and F_y^{i+1} for all $y \in I^{k-l}$:

$$G_y^i = \begin{cases} F_y^{i+1} & \text{on } U_i^{\text{top}}(y) \cap U_{i+1}^{\text{top}}, \\ F_y^{i-1} & \text{on } U_i^{\text{bot}}(y) \cap U_{i-1}^{\text{bot}}; \end{cases}$$

- $\|G_y^i - F_y^i\|_{C^0} < \varepsilon$ for all $y \in I^{k-l}$;
- $G_y^i|_{U_i^\partial(y)} = F_y^i|_{U_i^\partial(y)} = F|_{U_i^\partial(y)}$ for all $y \in I^{k-l}$;
- $G_y^i|_{U_i(y)} = F_y^i|_{U_i(y)} = F|_{U_i(y)}$ for all $y \in \mathcal{O}p \partial I^{k-l}$.

For even values $i = 0, 2, \dots, 2N$ we set $G_y^i \equiv F_y^i$. Take a cut-off function $\theta_N : \mathbb{R} \rightarrow I$, which is equal to 0 on $\mathcal{O}p(\mathbb{R} \setminus I)$ and is equal to 1 on $[\frac{1}{4N}, 1 - \frac{1}{4N}]$, and define a function $\varphi_N : \mathbb{R}^k \rightarrow \mathbb{R}$ by the formula

$$\varphi_N(y, t, x) = \theta_N(t)\theta_N(\|x\|)\theta_N(\|y\|)\cos 2N\pi t, \quad y \in \mathbb{R}^{k-l}, t \in \mathbb{R}, x \in \mathbb{R}^{l-1}.$$

Consider the map $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \delta_1 \varphi_N(x_1, \dots, x_k))$$

where $\delta/2 < \delta_1 < \delta$. Viewing the image $\tilde{I}^l(y) = h(I^l(y))$ as the union

$$\tilde{I}^l(y) = \bigcup_0^{2N-1} \tilde{I}_i(y),$$

where

$$\tilde{I}_i(y) = \tilde{I}^l(y) \cap \left\{ \frac{i}{2N} \leq t = x_{k-l+1} \leq \frac{i+1}{2N} \right\}$$

we define a continuous family of holonomic sections

$$\tilde{F}_y : \mathcal{O}p \tilde{I}^l(y) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q), \quad y \in I^{k-l}$$

by the formula

$$\tilde{F}_y|_{\mathcal{O}p \tilde{I}_i} = G_y^i|_{\mathcal{O}p \tilde{I}_i}, \quad i = 0, \dots, 2N-1.$$

Then the section $\tilde{F} : \mathcal{O}p h(I^k) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$ defined by the formula

$$\tilde{F}(y, t, x) = \tilde{F}_y(t, x) \quad \text{for } (y, t, x) \in \mathcal{O}p(I^{k-l} \times I \times I^{n-k-l-1})$$

is holonomic with respect to the fibration $\pi_l \circ h$ and is ε -close to F . ►

Proof of Theorem 1.3.1: We will prove the theorem by an induction over l . Consider for $l = 0, \dots, k$ the following

Inductional hypothesis $\mathcal{A}^{(l)}$. Let $F : \mathcal{O}p I^k \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$ be a section which is holonomic over $\mathcal{O}p \partial I^k$. For arbitrary small $\delta, \varepsilon > 0$ there exist a diffeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_k)),$$

and a section

$$\tilde{F}^l : \mathcal{O}p h(I^n) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

as required by the Inductional lemma, i.e

- $|\varphi| < \delta$; $\varphi|_{\mathcal{O}_p \partial I^k} = 0$;
- the image $h(I^k)$ is contained in the domain of the definition of the section F ;
- $\tilde{F}^l|_{\mathcal{O}_p \partial I^k} = F|_{\mathcal{O}_p \partial I^k}$;
- $\|\tilde{F}^l - F\|_{\mathcal{O}_p h(I^k)}|_{C^0} < \varepsilon$;
- the section $\tilde{F}^l|_{h(I^k)}$ is fiberwise holonomic with respect to the fibration

$$\pi_l \circ h^{-1} : h(I^k) \rightarrow I^{k-l}.$$

According to 1.1.1 the given section $F : I^n \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$, which is holonomic near ∂I^k , is tautologically fiberwise holonomic with respect to the fibration by points $\pi_0 : I^k \rightarrow I^k$. This implies $\mathcal{A}^{(0)}$ and thus gives us the base for the induction. For $l = 1$ the implication $\mathcal{A}^{(l-1)} \Rightarrow \mathcal{A}^{(l)}$ follows immediately from Inductional lemma 1.3.2, but in the general case $l > 1$ we cannot apply 1.3.2 directly because the section F^{l-1} is defined near the deformed cube rather than the original one. Notice however, that the diffeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ induces the covering map $h_* : J^r(\mathbb{R}^n, \mathbb{R}^q) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$. The section $\bar{F}^{l-1} = (h_*)^{-1}(\tilde{F}^{l-1})$ is defined over $\mathcal{O}_p I^k$, coincides with F near ∂I^k and fiberwise holonomic with respect to the fibration $\pi_{l-1} : I^k \rightarrow I^{k-l+1}$. Applying Inductional lemma 1.3.2 we can approximate \bar{F}^{l-1} by a section \tilde{F}' over a deformed cube $h'(I^k)$. The section \tilde{F}' coincides with \bar{F}^{l-1} near ∂I^k and fiberwise holonomic with respect to the fibration $\pi_l \circ h' : h'(I^k) \rightarrow I^{k-l}$. If \tilde{F}' is sufficiently C^0 -close to \bar{F}^{l-1} , then the section $\tilde{F}^l = h_*(\tilde{F}')$ is the required approximation of F in a neighborhood of $h''(I^k)$, where $h'' = h \circ (h')$. This proves $\mathcal{A}^{(l)}$ and Theorem 1.3.1. \blacktriangleright

Parametric case. It turns out that Inductional lemma 1.3.2 implies also the parametric version of Theorem 1.3.1. Namely, we have

1.3.3 (Parametric version of Theorem 1.3.1) *Let F_z , $z \in I^m$, be a family of sections $\mathcal{O}_p I^k \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$, parameterized by the cube I^m . Suppose that $k < n$ and the sections F_z are holonomic over $\mathcal{O}_p \partial I^k$ for all $z \in I^m$, and holonomic over the whole I^k for $z \in \mathcal{O}_p \partial I^m$. Then for arbitrary small $\delta, \varepsilon > 0$ there exist a family of diffeomorphisms $h_z : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form*

$$h_z(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi_z(x_1, \dots, x_k)),$$

and a family of holonomic sections

$$\tilde{F}_z : \mathcal{O}_p h_z(I^k) \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that

- $|\varphi_z| < \delta; \quad \varphi_z|_{\mathcal{O}_p \partial I^k} = 0;$
- the section F_z is defined in a neighborhood of $h_z(I^k)$ $z \in I^m;$
- $\tilde{F}_z|_{\mathcal{O}_p \partial I^k} = F_z|_{\mathcal{O}_p \partial I^k}$
- $\tilde{F}_z = F_z$ for $z \in \mathcal{O}_p I^m$ and
- $\|\tilde{F}_z - F_z|_{\mathcal{O}_p h_z(I^k)}\|_{C^0} < \varepsilon.$

Proof: Consider the cube $I^{k+m} = I^k \times I^m \subset \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+m}$. Let $J^r(\mathbb{R}^{n+m}|\mathbb{R}^n, \mathbb{R}^q)$ be the bundle whose restriction to $\mathbb{R}^n \times z$, $z \in \mathbb{R}^m$, equals $J^r(\mathbb{R}^n, \mathbb{R}^q)$. The family of sections

$$F_z : I^k \rightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

can be viewed as a section

$$\overline{F} : I^{k+m} \rightarrow J^r(\mathbb{R}^{n+m}|\mathbb{R}^n, \mathbb{R}^q).$$

The section \overline{F} lifts to a section $\overline{\overline{F}} : I^{k+m} \rightarrow J^r(\mathbb{R}^{n+m}, \mathbb{R}^q)$, so that $\pi \circ \overline{\overline{F}} = \overline{F}$, where

$$\pi : J^r(\mathbb{R}^{n+m}, \mathbb{R}^q) \rightarrow J^r(\mathbb{R}^{n+m}|\mathbb{R}^n, \mathbb{R}^q)$$

is the canonical projection. Moreover, the section $\overline{\overline{F}}$ can be chosen holonomic near ∂I^{k+m} . Hence we can apply Theorem 1.3.1 to get an ε -approximation $\tilde{\tilde{F}}$ of $\overline{\overline{F}}$ over a δ -displaced cube $h(I^{k+m})$. Then the composition $\tilde{F} = \pi \circ \tilde{\tilde{F}} : I^{k+m} \rightarrow J^r(\mathbb{R}^{n+m}|\mathbb{R}^n, \mathbb{R}^q)$ can be interpreted as the required family $\{\tilde{F}_z\}_{z \in I^m}$ of holonomic ε -approximations of the family $\{F_z\}$ near $\{h_z(I^k)\}$. \blacktriangleright

In the same way as Theorem 1.3.1 implies Theorem 1.2.1, i.e. via the induction over skeleta, Theorem 1.3.3 implies Theorem 1.2.2.

2 Applications

2.1 Gromov's h-principle for Diff V -invariant differential relations over open manifolds

Differential relations. A differential relation (or condition) imposed on sections $\varphi : V \rightarrow X$ of a fiber bundle $X \rightarrow V$ is a subset $\mathcal{R} \subset X^{(r)}$, where r is called the *order* of \mathcal{R} . A section $\Phi : V \rightarrow X^{(r)}$

is called a *formal solution* of \mathcal{R} if $\Phi(V) \subset \mathcal{R}$. A C^r -section $\varphi : V \rightarrow X$ is called a *solution* of the relation \mathcal{R} if $J_\varphi^r(V) \subset \mathcal{R}$. We will denote the space of solution of \mathcal{R} by $\text{Sol } \mathcal{R}$, the space of formal solutions by $\text{Sec } \mathcal{R}$, and the space of holonomic formal solutions by $\text{Hol } \mathcal{R}$. The r -jet extension establishes a one-to-one correspondence

$$J^r : \text{Sol } \mathcal{R} \rightarrow \text{Hol } \mathcal{R}$$

between the solutions and holonomic formal solutions of \mathcal{R} . We will use the term “solution” also for the sections from $\text{Hol } \mathcal{R}$ when the distinction between the solutions of \mathcal{R} as sections of X or $X^{(r)}$ is clear from the context, or irrelevant.

Diff V -invariant differential relations. Given a fibration $p : X \rightarrow V$ we will denote by $\text{Diff}_V X$ the group of fiber-preserving diffeomorphisms $h_X : X \rightarrow X$, i.e. $h_X \in \text{Diff}_V X$ if and only if there exists a diffeomorphism $h_V : V \rightarrow V$ such that $p \circ h_X = h_V \circ p$. Let $\pi : \text{Diff}_V X \rightarrow \text{Diff } V$ be the projection $h_X \mapsto h_V$. We are interested when this arrow can be reversed, i.e. *when there exists a homomorphism $j : \text{Diff } V \rightarrow \text{Diff}_V X$ such that $\pi \circ j = \text{id}$* . We call a fibration $X \rightarrow V$, together with a homomorphism j , *natural* if such a lift exists. For instance, the trivial fibration $X = V \times W \rightarrow V \rightarrow V$ is natural. Here $j(h_V) = h_V \times \text{id}$. The tangent bundle $T(V) \rightarrow V$ is natural as well. If a fibration $X \rightarrow V$ is natural then any fibration associated with it is natural as well. In particular, if $X \rightarrow V$ is natural then $X^{(r)} \rightarrow V$ is natural. The implied lift

$$j^r : \text{Diff } V \rightarrow \text{Diff}_V X^{(r)}, \quad h \mapsto h_*$$

is defined here by the formula

$$h_*(s) = J_{j(h) \circ \bar{s}}^r(h(v))$$

where $s \in X^{(r)}$, $v = p^r(s) \in V$, and \bar{s} is a local section near v which represents the r -jet s .

Given a natural fibration $X \rightarrow V$, a differential relation $\mathcal{R} \subset X^{(r)}$ is called *Diff V -invariant* if the action $s \mapsto h_*s$, $h \in \text{Diff } V$ leaves \mathcal{R} invariant. In other words, a differential relation \mathcal{R} is Diff V -invariant if it can be defined in V -coordinate free form. Notice that though the definition of a Diff V -invariant relation depends on the choice of the homomorphism j this choice is fairly obvious in most interesting examples, and we will not specify it.

The action $s \mapsto h_*s$ preserves the set of holonomic sections:

$$h_*(J_f^r) = J^r(j(h) \circ f \circ h^{-1}),$$

$f \in \text{Sec} X$, $h \in \text{Diff } V$. In particular, the group $\text{Diff } V$ acts on the space $\text{Sol } \mathcal{R}$ of an invariant differential relation \mathcal{R} .

Homotopy principle. Existence of a formal solution is a necessary condition for the solvability of a differential relation \mathcal{R} , and thus before trying to solve \mathcal{R} one should check whether \mathcal{R} admits a formal solution. The problem of finding formal solutions is of pure homotopy-theoretical nature. This problem maybe simple, or highly non-trivial, but in any case it is important to treat the homotopical problem first, and look for genuine solutions only after existence of formal solutions have been already established.

Though finding a formal solution is an algebraic, or homotopy-theoretical problem which is a dramatic simplification of the original differential problem, and at first thought the existence of a formal solution cannot be sufficient for the genuine solvability of \mathcal{R} , in the second half of XX century there were discovered many large and geometrically interesting classes of differential relations \mathcal{R} for which the solvability of the formal problem turned out to be sufficient for the genuine solvability. Moreover, in most of these examples the spaces of formal and genuine solutions are much closer related that one could expect.

A differential relation \mathcal{R} is said to satisfy the *(parametric) h-principle*, or *homotopy principle* if the inclusion $\text{Hol } \mathcal{R} \hookrightarrow \text{Sec } \mathcal{R}$ is a weak homotopy equivalence. This means, in particular, that any formal solution can be deformed inside \mathcal{F} into a genuine solution and any two solutions which can be joined by a family of formal solutions can be also joined by a family of genuine solutions. Moreover, similar properties hold for families of formal and genuine solutions, depending on arbitrary number of parameters.

In fact, it is useful to consider different “degrees” of h -principles, when one want to establish closer and closer connection between formal and genuine solutions. For instance, different forms of h -principles may include some approximation and extension properties.

Open manifolds. A manifold V is called *open*, if there are no closed manifolds among its connected components. As it is well known

2.1.1 *If V is open, then there exists a polyhedron $K \subset V$, $\text{codim } V_0 \geq 1$, such that V can be compressed by an isotopy $g^t : V \rightarrow V$, $t \in [0, 1]$, into an arbitrary small neighborhood U of V_0 .*

Here is the main theorem of this section.

2.1.2 (Gromov, 1968) *Let V be an open manifold and $X \rightarrow V$ a natural fiber bundle. Then any open Diff V -invariant differential relation $\mathcal{R} \subset X^{(r)}$ satisfies the parametric h -principle.*

Proof: We need to proof that given a family of sections $F_z \in \text{Sec } \mathcal{R}$, $z \in I^m$, $m = 0, \dots$, such that for $z \in \mathcal{O}p \partial I^m$ the section F_z is holonomic, there exists a family $\tilde{F}_z \in \text{Hol } \mathcal{R}$ which is homotopic in $\text{Sec } \mathcal{R}$ to the family F_z , $z \in I^m$, relative to ∂I^m . Let $K \subset V$ be a polyhedron of positive codimension, as in 2.1.1. According to Theorem 1.2.2 there exist a C^0 -small family of diffeomorphisms $h_z : V \rightarrow V$ and a family $G_z \in \text{Hol}_{\mathcal{O}p h_z(K)} \mathcal{R}$, such that $h_z = \text{Id}$ for $z \in \partial I^m$, and G_z is C^0 -close to F_z over $\mathcal{O}p h_z(K)$. Let U be a small neighborhood of K , such that for all $z \in I^m$ the image $h_z(U)$ is contained in the domain of the definition of the section G_z . Let $g^t : V \rightarrow V$, $t \in [0, 1]$, be the isotopy compressing V into the neighborhood U . Then for every $z \in I^m$ the isotopy $g_z^t = h_z \circ g^t \circ h_z^{-1}$, $t \in [0, 1]$, compresses V into a neighborhood of $h_z(K)$ where the section G_z is defined. The desired family of sections $\tilde{F}_z \in \text{Hol } \mathcal{R}$ can be now defined by the formula

$$\tilde{F}_z = (g_z^1)_*^{-1}(G_z), \quad z \in I^m,$$

where $(g_z^1)_* : X^{(r)} \rightarrow X^{(r)}$ is the induced action of the diffeomorphism g_z^1 on the natural fibration $X^{(r)} \rightarrow V$. ►

Notice, that the C^0 -dense h -principle for \mathcal{R} does not necessarily hold, though the C^0 -dense h -principle holds for \mathcal{R} near A . The relative h -principle does not hold either. However, we have its following version.

2.1.3 *Let $\mathcal{R} \subset X^{(r)}$ be an open Diff V -differential relation over an open manifold V . Let $B \subset V$ be a closed subset such that each connected component of the complement $V \setminus B$ has a boundary point which is not in B . Then the relative parametric h -principle holds for \mathcal{R} and the pair (V, B) .*

2.2 Directed embeddings of open manifolds

Let $Gr_n(W)$ be the Grassmanian bundle of tangent n -planes to a q -dimensional manifold W , $q > n$, and $A \subset Gr_n(W)$ be an arbitrary subset. A map $f : V \rightarrow W$ of a n -dimensional manifold V is called A -directed if the tangential (Gauss) map $G_f : V \rightarrow Gr_n(W)$ sends V into A . Any A -directed map is automatically an immersion. If A is open, then the h -principle for A -directed maps $V \rightarrow W$ follows immediately from 2.1.2, since the corresponding differential relation \mathcal{R}_A is open and Diff V -invariant. For A -directed embeddings Gromov proved in [G2] via his convex integration technique

the following ²

2.2.1 (Homotopy principle over embeddings) *If $A \subset Gr_n(W)$ is open and V is an open manifold, then every embedding $f_0 : V \rightarrow W$, whose tangential lift $G_0 = G_{f_0} : V \rightarrow Gr_n(W)$ is homotopic over embeddings to a map $G_1 : V \rightarrow A \subset Gr_n(W)$ can be isotoped to a A -directed embedding $f_1 : V \rightarrow W$.*

Here the homotopy *over embeddings* means that the underlying homotopy $g_t : V \rightarrow W$ is an isotopy. We show below how Theorem 2.2.1 can be deduced from Holonomic approximation theorem 1.2.1.

Proof: Suppose that the homotopy G_t is fixed on the base (that is the underlying isotopy $g_t : V \rightarrow W$ is fixed on V) and *small* in the following sense: the angle between $T_v(V)$ and $G_t(T_v(V))$ is less then, say, $\pi/4$ for all $v \in V$ and $t \in [0, 1]$. Let $K \subset V$ be a codimension ≥ 1 compact subcomplex in V , such that V can be compressed into an arbitrary small neighborhood of K . The plane fields $G_t(T(V))$ along V define a homotopy of section $F_t : V \rightarrow X^{(1)}$ of the fibration $X^{(1)} \rightarrow V$, where X is a tubular neighborhood of $V \subset W$. Using Theorem 1.2.1 we can construct a holonomic approximation \tilde{F}_1 of F_1 near $h_1(K)$, where $h_t : V \rightarrow V$ is a diffeotopy. For a sufficiently close approximation the section \tilde{F}_1 over $h_1(K)$ still lies in (the open set) A . Therefore the graph of the 0-jet part of this new section over a neighborhood of $h_1(K) \subset V$ is the image of the required embedding $f_1 : V \rightarrow W$. The respective isotopy f_t is the composition of the compression and the isotopy of sections.

Any fixed on V (non-small) homotopy G_t may be decomposed into a finite sequence of some small homotopies. Therefore we can apply the previous construction successively and get the required isotopy f_t .

If the general case when the homotopy G_t is not fixed on V we may apply the previous construction to the pull-back homotopy $(d\varphi_t)^{-1} \circ G_t$ and the pull-back relation $\mathcal{R}_{\tilde{A}} = \varphi_1^*(\mathcal{R}_A)$, where $\varphi_t : W \rightarrow W$ is a diffeotopy, which extends the underlying for G_t isotopy $g_t : V \rightarrow W$. ►

The parametric h -principle is also valid in this case (with the same proof).

² Gromov's proof is discussed in [Sp]. Rourke and Sanderson gave two independent proofs of this theorem in [RS1] and [RS2]. The proof in [RS2] is quite close to ours.

2.3 Final remarks

Geometrically interesting examples covered by Theorem 2.1.2 include (see [G2]) in the case of open manifolds immersions, submersions, k -mersions (i.e. mappings of rank $\geq k$), mappings with non-degenerate higher-order osculating spaces, mappings transversal to foliations, or more generally, to arbitrary distributions, construction of generating system of exact differential k -forms, symplectic and contact structures on open manifolds, etc.

On the other hand, the *micro-extension* trick which goes back to M. Hirsch, allows sometimes to reformulate the problems about closed manifolds in terms of open manifolds. For example, if $\dim W > \dim V$ then a construction of an immersion $V \rightarrow W$ homotopic to a map $f : V \rightarrow W$ is equivalent to a construction of an immersion $E \rightarrow W$, where E is the total space of the normal bundle to $T(V)$ in $f^*T(W)$. The manifold E is open and hence the h -principle 2.1.2 applies. The same trick yields in some cases the directed embedding theorem for closed manifolds. For instance, Theorem 2.2.1 allows, when the necessary homotopical conditions are met, to perturb a closed n -dimensional submanifold $V \subset \mathbb{R}^q$ via an isotopy so that its projection to \mathbb{R}^k , where $n < k < q$, becomes an immersion.

Gromov showed in [G1] that the relative parametric h -principle is equivalent to the relative h -principle for foliated manifolds. For instance, a manifold V with a k -dimensional foliation \mathcal{F} , such that the tangent to the foliation bundle $\xi = T(\mathcal{F})$ is trivial, admits a map to \mathbb{R}^{k+1} whose restriction to each leaf is an immersion. A similar statement for a non-integrable sub-bundle $\xi \subset T(V)$ does not follow from Theorem 2.1.2. To prove this and some other geometric corollaries one needs several modifications of the h -principle 2.1.2, which can be found in Gromov's book [G2]. One of these modifications is concerned with the differential relation invariant with respect to certain subgroups of $\text{Diff } V$. For instance, the statement about mappings non-degenerate along (not necessarily integrable) tangent distributions requires the following theorem from Gromov's book [G2].

2.3.1 *Let $X \rightarrow V \times \mathbb{R}$ be a natural fibration and $\mathcal{R} \subset X^{(r)}$ be an open differential relation invariant with respect to diffeomorphisms of the form*

$$(x, t) \mapsto (x, h(t)) \quad x \in V, t \in \mathbb{R}.$$

Then \mathcal{R} satisfies the parametric h -principle.

Proposition 2.3.1 is also a corollary of our Theorem 1.3.1. The proof follows the same scheme as the proof of 2.1.2 with an additional remark that the perturbation h implied by 1.3.1 has a special form precisely as Proposition 2.3.1 requires.

Some other important geometric applications in Gromov's book, e.g. symplectic and contact embeddings theorems, need a generalization of Theorem 2.1.2 from open to *micro-flexible* differential relations. We will not discuss here the notion of micro-flexibility and only note that this property is equivalent to the Interpolation Property considered in the proof of Theorem 1.3.1. Hence it is straightforward to adjust Theorem 1.2.1 so that it would serve the needs of the micro-flexible case. We refer the reader to our forthcoming book [EM] for the details and further applications of Holonomic Approximation Theorem.

References

- [EM] Y. Eliashberg and N.M. Mishachev, *Flexible integration. Introduction to h -principle*, in preparation
- [G1] M. Gromov, *Stable maps of foliations into manifolds*, Izv. AN SSSR, ser. mat. **33**(1969), 707–734.
- [G2] M. Gromov, *Partial differential relations*, Springer-Verlag, 1986
- [H] A. Haefliger, *Lectures on the theorem of Gromov*, Lecture Notes in Math., Vol. **209**(1971), pp. 128–141
- [Hi] M. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93**(1959), 242–276
- [Ph] A. Phillips, *Submersions of open manifolds*, Topology **6**(1967), 171–206
- [RS1] C. Rourke, B. Sanderson, *The compression theorem* (1997), preprint
- [RS2] C. Rourke, B. Sanderson, *Directed embeddings: a short proof of Gromov's theorem* (2000), preprint
- [Sm] S. Smale, *The classification of immersions of spheres in Euclidean spaces*, Ann. of Math. (2) **69**(1959), 327–344
- [Sp] D. Spring, *Directed embeddings and the simplification of singularities* (2000), preprint